

Planck Scale Effects in Electrodynamics of a Generalized Charged Particle

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Abstract:

Extensions of free relativistic particle models, living in Non-Commutative space-times that might be relevant in the context of Quantum Gravity in Planck scale physics are well known. In the present work the dynamics of such charged particles in the presence of external electromagnetic interactions are studied in detail. Furthermore we have also considered the electromagnetic field to be dynamical and have derived the modified forms of Lienard-Wiechert like potentials for these extended charged particle models. In all the above cases we exploit the new form of κ -Minkowski algebra where electromagnetic effects are incorporated in the lowest order, in the Dirac framework of Hamiltonian constraint analysis.

I. INTRODUCTION

From a theoretical perspective, it appears that the existing theoretical framework of physical laws might require a significant modification in the regime of Quantum Gravity that is for Planck scale physics. A particularly interesting scenario, termed as Doubly Special Relativity (DSR), was pioneered by Amelino-Camelia [1]. (For reviews see for example [2]). This model is an extension of Einstein's Special Relativity.

In Special Relativity there is one (inertial) observer independent dimensional parameter, the velocity of light but in DSR there are two observer independent dimensional parameters, the velocity of light and an energy scale, conveniently chosen as the Planck energy. Since it is expected that Planck scale provides the borderline of qualitatively different physics, (related to the nature of spacetime in particular), it is intuitively obvious that all inertial observers should agree on the same value of the Planck scale. The mathematical framework of DSR appears most natural in Non-Commutative (NC) geometry [2, 3] and a well studied NC phase space is the one obeying κ -Minkowski NC algebra [4]. The NC geometry was introduced in physics very early by [5] and resurrected to its present interest by the work of [6]. Out of several alternative bases of the algebra [3], that are inter-related through non-linear transformations, we will concentrate on the Magueijo-Smolín (MS) base [7]. In general, extended relativistic principles in DSR requires a modification in the Einstein energy-momentum relation for a free particle of rest mass m_0 and momentum P_μ , $P_\mu P^\mu = m_0^2$. Quite naturally the canonical Lagrangian $L = m_0 \sqrt{((dX_\mu)/(d\tau))^2}$ also needs to be drastically altered to account for the DSR particle dispersion relation and κ -Minkowski NC phase space. Two of the present authors [8], along with many other workers [9], have provided explicit models for DSR particles that is consistent with the κ -Minkowski NC algebra and DSR dispersion relation. Many novel features of the DSR particle have come in to light from the analysis of these models.

It needs to be emphasized that mostly these works deal with the free DSR particle but many interesting properties as well as drawbacks of the model (if there are any) are not encountered unless the particle is subjected to non-trivial interactions. In the present paper we have precisely done that. We have considered a DSR particle interacting with a $U(1)$ gauge theory. We have mostly concentrated on treating the gauge field as external but we have also commented on the DSR effects on the Lienard-Wichert type potentials arising from the charged DSR particle.

The paper is organized as follows: in Section II we introduce the κ -Minkowski NC algebra in MS base and the interacting DSR particle - $U(1)$ gauge theory. In Section III we derive the extended κ -Minkowski algebra in presence of $U(1)$ gauge interaction and obtain the equations of motion. Section IV deals with explicit solutions for particle trajectories in a perturbative framework that provide valuable insights of the DSR effects on a charged particle motion. In Section V we comment on the electromagnetic potentials induced by the charged DSR particle. The paper concludes in Section VI with discussions and future outlook.

II. INTERACTING CHARGED MS PARTICLE IN $U(1)$ FIELD

The κ -Minkowski NC phase space algebra is defined as,

$$\{x^i, x^0\} = \frac{x^i}{\kappa} ; \quad \{x^i, x^j\} = 0 ; \quad \{x^i, p^j\} = -g^{ij} ; \quad \{p^\mu, p^\nu\} = 0. \quad (1)$$

Our metric is $\text{diag } g^{00} = -g^{ii} = 1$ and κ is the NC parameter. Rest of the phase space algebra is given below,

$$\{x^0, p^i\} = p^i/\kappa ; \{x^i, p^0\} = 0 ; \{x^0, p^0\} = -1 + p^0/\kappa. \quad (2)$$

The above is rewritten in a covariant form,

$$\begin{aligned} \{x_\mu, x_\nu\} &= \frac{1}{\kappa}(x_\mu \eta_\nu - x_\nu \eta_\mu), \\ \{x_\mu, p_\nu\} &= -g_{\mu\nu} + \frac{1}{\kappa}\eta_\mu p_\nu, \quad \{p_\mu, p_\nu\} = 0, \end{aligned} \quad (3)$$

where $\eta_0 = 1, \eta_i = 0$. This algebra appeared in [10]. Detailed studies of similar types of algebra are provided in [3]. This algebra has emerged before in an earlier work of one of us [11] where it was embedded in a more general algebra. For $\kappa \rightarrow \infty$ one recovers the normal canonical phase space. Using κ -Minkowski NC algebra (3), it is straightforward to check that the Lorentz generators $J_{\mu\nu} = x_\mu p_\nu - x_\nu p_\mu$ satisfy the canonical Lorentz algebra $J^{\mu\nu}$,

$$\{J^{\mu\nu}, J^{\alpha\beta}\} = g^{\mu\beta} J^{\nu\alpha} + g^{\mu\alpha} J^{\beta\nu} + g^{\nu\beta} J^{\alpha\mu} + g^{\nu\alpha} J^{\mu\beta}. \quad (4)$$

However the particle dispersion relation that remains invariant under (3) with respect to $J^{\mu\nu}$ treated as rotation generators in the NC framework is given below:

$$p^2 = m^2 \left[1 - \frac{(\eta p)}{\kappa} \right]^2 = m^2 \left[1 - \frac{E}{\kappa} \right]^2. \quad (5)$$

We use the shorthand notation $(ab) \equiv a^\mu b_\mu$. This is the well known MS dispersion relation that reduces to the Special Theory relation for $\kappa \rightarrow \infty$.

In order to derive a Lagrangian for the DSR particle *ab initio* it is quite difficult to guess how to proceed since it has to be compatible with the complicated κ -Minkowski symplectic structure (3). In this perspective, it is very convenient to exploit the Darboux-like canonical variables. The idea is to construct the Lagrangian in terms of canonical variables which are in fact the Darboux map of the physical non-canonical (or NC) degrees of freedom. The next step is to reexpress the the Lagrangian in term of the physical NC variables via the inverse Darboux map. We have successfully exploited this technique in a previous work [8]. In the present case, with the canonical phase space,

$$\{X_\mu, P_\nu\} = -g_{\mu\nu}; \quad \{X_\mu, X_\nu\} = \{P_\mu, P_\nu\} = 0, \quad (6)$$

the inverse Darboux map is

$$X_\mu \equiv x_\mu \left(1 - \frac{(\eta p)}{\kappa} \right) = x_\mu \left(1 - \frac{E}{\kappa} \right); \quad P_\mu \equiv \frac{p_\mu}{\left(1 - \frac{(\eta p)}{\kappa} \right)} = \frac{p_\mu}{\left(1 - \frac{E}{\kappa} \right)}, \quad (7)$$

whereas the Darboux map is,

$$x_\mu = X_\mu \left(1 + \frac{(\eta P)}{\kappa} \right) = X_\mu \left(1 + \frac{P_0}{\kappa} \right); \quad p_\mu = \frac{P_\mu}{\left(1 + \frac{(\eta P)}{\kappa} \right)} = \frac{P_\mu}{\left(1 + \frac{P_0}{\kappa} \right)}. \quad (8)$$

We consider the first order form of a particle interacting minimally with $U(1)$ gauge field,

$$\begin{aligned} L &= P^\mu(\tau) \dot{X}_\mu(\tau) - \lambda(P^2(\tau) - m_0^2) + \int d^4y \, e \, \delta(y - X(\tau)) \dot{X}_\mu A^\mu(y) \\ &= P^\mu(\tau) \dot{X}_\mu(\tau) - \lambda(P^2(\tau) - m^2) + e \, X_\mu \dot{X}_\mu A^\mu(X(\tau)) \\ &= (p\dot{x}) - \frac{(px)(\eta\dot{p})}{\kappa(1 - \frac{(\eta p)}{\kappa})} - \frac{\lambda}{2} \left(p^2 - m^2 \left(1 - \frac{(\eta p)}{\kappa} \right)^2 \right) - e A^\mu(X) \left(\dot{x}_\mu \left(1 - \frac{(\eta p)}{\kappa} \right) - x_\mu \frac{(\eta\dot{p})}{\kappa} \right). \end{aligned} \quad (9)$$

In the last equality we have used the inverse Darboux map (7) and the notation $a^\mu b_\mu \equiv (ab)$. As has been shown before [8], for the free theory $e = 0$, Dirac constraint analysis [12] yields the NC κ -Minkowski symplectic structure.

III. NONCOMMUTATIVE SPACE AND DYNAMICS OF INTERACTING MS PARTICLE IN EXTERNAL $U(1)$ FIELD

Our first task is to compute the symplectic structure modified by the electromagnetic interaction. We utilize the Dirac formalism [12] for Hamiltonian analysis of constraint system. A similar analysis for the free theory, ($e = 0$) was done in [8]. From (II) the constraints are,

$$\eta_\mu^1 \equiv \Pi_\mu^x - p_\mu - e\psi A_\mu \approx 0, \quad (10)$$

$$\eta_\mu^2 \equiv \Pi_\mu^p + \frac{(px)\eta_\mu}{\kappa\psi} + \frac{e(xA)\eta_\mu}{\kappa} \approx 0, \quad (11)$$

where $\psi \equiv 1 - \frac{(\eta p)}{\kappa}$ and the conjugate momenta are

$$\Pi_\mu^x = \frac{\partial L}{\partial \dot{x}_\mu} = p_\mu + e\psi A_\mu; \quad \Pi_\mu^p = \frac{\partial L}{\partial \dot{p}_\mu} = -\frac{(px)\eta_\mu}{\kappa\psi} - \frac{e(xA)\eta_\mu}{\kappa}. \quad (12)$$

The constraints η_μ^1, η_μ^2 are non-commuting since

$$\{\eta_\mu^1, \eta_\nu^1\} = e\psi^2 \left(\frac{\partial A_\mu}{\partial X_\nu} - \frac{\partial A_\nu}{\partial X_\mu} \right) = e\psi^2 F_{\mu\nu}, \quad (13)$$

$$\{\eta_\mu^1, \eta_\nu^2\} = g_{\mu\nu} + \frac{p_\mu \eta_\nu}{\kappa\psi} + \frac{e\psi}{\kappa} F_{\mu\lambda} x^\lambda \eta_\nu \quad (14)$$

$$\{\eta_\mu^2, \eta_\nu^2\} = \frac{(x_\mu \eta_\nu - x_\nu \eta_\mu)}{\kappa \psi}. \quad (15)$$

Hence the constraints are Second Class in Dirac sense and we wish to compute the Dirac bracket [12], as defined below,

$$\{A, B\}^* = \{A, B\} - \{A, \eta^i\} \{\eta^i, \eta^j\}^{-1} \{\eta^j, B\}, \quad (16)$$

so that the constraints can be strongly put to zero. Expressing the constraint matrix as

$$\{\eta_\mu^i, \eta_\nu^j\} = \begin{bmatrix} 0 & g_{\mu\nu} + \frac{p_\mu \eta_\nu}{\kappa \psi} \\ -g_{\mu\nu} - \frac{p_\nu \eta_\mu}{\kappa \psi} & \frac{(x_\mu \eta_\nu - x_\nu \eta_\mu)}{\kappa \psi} \end{bmatrix} + e \begin{bmatrix} \psi^2 F_{\mu\nu} & \frac{\psi}{\kappa} F_{\mu\lambda} x^\lambda \eta_\nu \\ -\frac{\psi}{\kappa} F_{\nu\lambda} x^\lambda \eta_\mu & 0 \end{bmatrix} \equiv A + eB \quad (17)$$

and using the inverse

$$(A + eB)^{-1} = A^{-1} - e(A^{-1}BA^{-1}) + O(e^2)$$

the required inverse of the constraint matrix to $O(e)$ is computed below:

$$\{\eta_\nu^i, \eta_\lambda^j\}^{-1} = \begin{bmatrix} \frac{1}{\kappa}(x_\nu \eta_\lambda - x_\lambda \eta_\nu) & -g_{\nu\lambda} + \frac{1}{\kappa} \eta_\nu p_\lambda + \frac{e\psi^2}{\kappa} x_\nu \eta^\alpha F_{\alpha\lambda} \\ g_{\nu\lambda} - \frac{1}{\kappa} \eta_\lambda p_\nu - \frac{e\psi^2}{\kappa} x_\lambda \eta^\alpha F_{\alpha\nu} & e\psi^2 [F_{\nu\lambda} + \frac{\eta^\alpha}{\kappa} (p_\nu F_{\alpha\lambda} - p_\lambda F_{\alpha\nu})] \end{bmatrix}. \quad (18)$$

Therefore to $O(e)$ the Dirac Brackets are

$$\{x_\alpha, x_\beta\}^* = \{x_\alpha, x_\beta\} - \{x_\alpha, \eta_\nu^1\} \{\eta_\nu^1, \eta_\lambda^1\}^{-1} \{\eta_\lambda^1, x_\beta\} = \frac{1}{\kappa} (x_\alpha \eta_\beta - x_\beta \eta_\alpha) \quad (19)$$

$$\{x_\alpha, p_\beta\}^* = \{x_\alpha, p_\beta\} - \{x_\alpha, \eta_\nu^1\} \{\eta_\nu^1, \eta_\lambda^2\}^{-1} \{\eta_\lambda^2, p_\beta\} = (-g_{\alpha\beta} + \frac{1}{\kappa} \eta_\alpha p_\beta) + \frac{e\psi^2}{\kappa} x_\alpha \eta^\nu F_{\nu\beta} \quad (20)$$

$$\{p_\alpha, p_\beta\}^* = \{p_\alpha, p_\beta\} - \{p_\alpha, \eta_\nu^2\} \{\eta_\nu^2, \eta_\lambda^2\}^{-1} \{\eta_\lambda^2, p_\beta\} = e\psi^2 (F_{\alpha\beta} + \frac{\eta^\nu}{\kappa} (p_\alpha F_{\nu\beta} - p_\beta F_{\nu\alpha})). \quad (21)$$

For $e = 0$ the above reduces to the free κ -Minkowski algebra [3]. It is worth pointing out that $\{x_\alpha, x_\beta\}$ remains unchanged and the modifications involve the gauge invariant field tensor $F_{\mu\nu}(X)$.

To obtain the equations of motion we write the Hamiltonian first in terms of canonical variables and subsequently in terms of NC variables:

$$H = \frac{P^2}{m_0} - \sqrt{P^2} = \frac{p^2}{m_0 \psi^2} - \frac{\sqrt{p^2}}{\psi} \quad (22)$$

As $P^2 = m_0^2 \Rightarrow p^2 = m_0^2 \psi^2$. Note that this H should not be identified as the energy but it is easy to check that in the canonical sector it clearly reproduces the correct Hamiltonian equations of motion,

$$(dx^\alpha)/d\tau \equiv \dot{x}^\alpha = \{x^\alpha, H\}^*, \quad \dot{p}^\alpha = \{p^\alpha, H\}^*. \quad (23)$$

We should mention that there is an approximation involved in deriving H in (22). Strictly speaking we should have used the Darboux map pertaining to the $U(1)$ modified algebra (19 - 21) whereas, mainly for simplicity we have used the Darboux map (7,8) for the free NC algebra (3) as derived in [8]. Although $O(e)$ correction to the Darboux map is straightforward to compute we have not considered this here. But as we demonstrate in rest of the paper, even in this relatively simplified situation, the κ -corrections generate strikingly qualitative differences.

Therefore the equations of motion to $O(e)$ follow:

$$\dot{x}^\alpha = (2 - \psi) \frac{1}{m_0} \left[-\frac{1}{\psi^2} p^\alpha + \frac{e}{\kappa} (\eta F p) x^\alpha \right], \quad (24)$$

$$\dot{p}^\alpha = (2 - \psi) \frac{e}{m_0} \left[F^{\alpha\beta} p_\beta + \frac{1}{\kappa} (\eta F p) p^\alpha + 2m_0^2 F^{\alpha\beta} \eta_\beta \right]. \quad (25)$$

Let us write down the modified Lorentz force equation to order $O(e)$. We differentiate (24) with respect to τ to obtain,

$$\ddot{x}^\alpha = (2 - \psi) \frac{1}{m_0} \left[-\frac{1}{\psi^2} \dot{p}^\alpha + \frac{e}{\kappa} (\eta F p) \dot{x}^\alpha + \frac{2\dot{\psi}}{m_0 \psi^3} p^\alpha \right] + \frac{\dot{\psi}}{m_0 \psi^2} p^\alpha. \quad (26)$$

Now using (24) we replace all p -dependent terms in the R.H.S. of (26) and keep only $O(e)$ terms through. This exercise yields,

$$m_0 \ddot{x}^\alpha = e F^{\alpha\beta} p_\beta + \frac{e}{\kappa} \left[-\frac{2(2 - \psi)^2 m_0}{\psi^2} F^{\alpha\beta} \eta_\beta + \frac{(4 - \psi) \psi^4 m_0}{(2 - \psi)} (\eta F \dot{x}) \left(\frac{\psi^2 m_0}{(2 - \psi)} (\eta \dot{x}) - 1 \right) \dot{x}^\alpha - \psi^2 m_0 (\eta F \dot{x}) \dot{x}^\alpha \right]. \quad (27)$$

Clearly the MS Lorentz force law is quite involved with non-linear terms coming in through the κ -correction and for $\kappa \rightarrow \infty$ the conventional Lorentz law is smoothly recovered.

IV. TRAJECTORIES OF CHARGED MS PARTICLES IN EXTERNAL $U(1)$ FIELD

We will try to solve perturbatively the first order τ -derivative force equation (25) for some specific external field configuration. Due to the nature of the basic (MS form of κ -Minkowski algebra in (3)) NC structure considered, it is easy to see that the κ -corrections always involve the electric field in the form $(\eta F)_\alpha$, as is clear from (19-21). Hence for situations with external purely magnetic fields the dynamics does not change very much in a qualitative way. Hence for the present case, we consider only external constant electric field. Again this analysis is true for combined electric and magnetic fields but with the condition $|\vec{E}| > |\vec{B}|$ such that one can Lorentz transform to another frame with electric field only (see for example [13]). Without loss of generality, let us consider the electric field to be along the direction of x_1 -axis, $E_1 = E, E_2 = E_3 = 0$. Breaking up the set of equations in (25), we find

$$\frac{dp_0}{d\tau} = \frac{eE}{m} p_1 + \frac{2eE}{m\kappa} p_0 p_1,$$

$$\begin{aligned}\frac{dp_1}{d\tau} &= \frac{eE}{m}p_0 + \frac{eE}{m\kappa}(p_0^2 + 2m^2 - p_1^2), \\ \frac{dp_2}{d\tau} &= -\frac{eE}{m\kappa}p_1p_2, \quad \frac{dp_3}{d\tau} = -\frac{eE}{m\kappa}p_1p_3.\end{aligned}\tag{28}$$

We immediately notice a qualitative difference: For the normal particle the force is impressed only along the direction of the external electric field and the energy changes accordingly. As for example in the present case the electric field is along x_1 and for the normal particle the momentum will change only x_1 direction. However, for the MS particle we observe that not only are the rates of changes energy and x_1 -momentum modified, but more interestingly momenta along x_2 and x_3 directions are also affected. Note that the mass m will actually depend on the initial velocity as we specify below. We further differentiate the above set of equation and keeping terms up to the first order term of $\frac{1}{\kappa}$, a little rearrangement leads to:

$$\frac{d^2p_0}{d\tau^2} = \epsilon^2 \left[p_0 + \frac{1}{\kappa} (3p_0^2 + p_1^2 + 2m^2) \right] \tag{29}$$

$$\frac{d^2p_1}{d\tau^2} = \epsilon^2 \left[p_1 + \frac{2}{\kappa} p_0 p_1 \right] \tag{30}$$

$$\frac{d^2p_2}{d\tau^2} = -\frac{\epsilon^2}{\kappa} p_0 p_2 \tag{31}$$

$$\frac{d^2p_3}{d\tau^2} = -\frac{\epsilon^2}{\kappa} p_0 p_3 \tag{32}$$

where $\frac{eE}{m} = \epsilon$.

This is the system of non-linear ordinary differential equation we plan to solve to the leading order in $1/\kappa$. Let the initial conditions for the momenta at $\tau = 0$ be $p_0(0) = m$, $p_2(0) = mv_0$, $p_1(0) = p_3(0) = 0$ so that $m = \frac{m_0}{\sqrt{1-V_0^2}}$, $v_0 = \frac{V_0}{\sqrt{1-V_0^2}}$ where $V_0 = \frac{dx}{dt}|_{t=0}$, $v_0 = \frac{dx}{d\tau}|_{\tau=0}$. We provide some details of solving perturbatively the non-linear set of equations (29-32) in the Appendix. To $O(1/\kappa)$ the solutions are,

$$\begin{aligned}p_0 &= m \cosh(\epsilon\tau) + \frac{2}{\kappa} \left(\frac{m^2}{6} + m^2 \right) \cosh(\epsilon\tau) + \frac{2m^2}{3\kappa} \cosh(2\epsilon\tau) - \frac{m^2 + 2m^2}{\kappa}, \\ p_1 &= m \sinh(\epsilon\tau) + \frac{2}{\kappa} \left(\frac{m^2}{6} + m^2 \right) \sinh(\epsilon\tau) + \frac{m^2}{3\kappa} \sinh(2\epsilon\tau), \\ p_2 &= mv_0 \left[1 + \frac{m}{\kappa} (1 - \cosh(\epsilon\tau)) \right], \quad p_3 = 0.\end{aligned}\tag{33}$$

Since $p_0 = m \frac{dx_0}{d\tau}$ and $p_i = m \frac{dx_i}{d\tau}$, integrating p_μ from the above set of equation with respect to τ gives x_μ as

$$m\epsilon x_0 = \left(m + \frac{m^2}{3\kappa} + \frac{2m^2}{\kappa}\right) \text{Sinh}(\epsilon\tau) + \frac{m^2}{3\kappa} \text{Sinh}(2\epsilon\tau) - \frac{(m^2 + 2m^2)\epsilon\tau}{\kappa}, \quad (34)$$

$$m\epsilon x_1 = \left(m + \frac{m^2}{3\kappa} + \frac{2m^2}{\kappa}\right) \text{Cosh}(\epsilon\tau) + \frac{m^2}{6\kappa} \text{Cosh}(2\epsilon\tau) - \left(m + \frac{m^2}{2\kappa} + \frac{2m^2}{\kappa}\right), \quad (35)$$

$$m\epsilon x_2 = mv_0 \left[\epsilon\tau \left(1 + \frac{m}{\kappa}\right) - \frac{m}{\kappa} \text{Sinh}(\epsilon\tau) \right], \quad (36)$$

$$x_3 = 0. \quad (37)$$

Hence we have found the paths of the MS particle in proper time. Finally we can eliminate τ from (35) and (36) to construct the trajectories of the particle as,

$$\left\{ x_1 + \frac{m}{eE} \left(1 + \frac{5m}{2\kappa}\right) \right\}^2 - \frac{x_2^2}{v_0^2} \left(1 + \frac{11m}{2\kappa}\right) - \frac{x_2^4}{v_0^4} \frac{e^2 E^2}{3m\kappa} = \frac{m^2}{e^2 E^2} \left(1 + \frac{5m}{\kappa}\right). \quad (38)$$

The κ -corrected hyperbolic path is recovered provided we neglect the x_2^4 term as it is of $O(e^2/\kappa)$. This leads to a modified hyperbola:

$$\left\{ \frac{eE}{m} x_1 + \left(1 + \frac{5m}{2\kappa}\right) \right\}^2 - \frac{e^2 E^2}{m^2 v_0^2} \left(1 + \frac{11m}{2\kappa}\right) x_2^2 = \left(1 + \frac{5m}{\kappa}\right) \quad (39)$$

The eccentricity Σ is reduced by the κ -correction,

$$\Sigma = \sqrt{1 + v_0^2 - \frac{11mv_0^2}{2\kappa}}. \quad (40)$$

Various features of the paths and trajectories of the MS particle are graphically shown in Section VI.

V. LIENARD-WIECHERT POTENTIAL FOR MS PARTICLE

The full interacting action for electrodynamics of the conventional particle is

$$\begin{aligned} A &= \int [-(d\tau) \sqrt{(\dot{X}^\mu(\tau) \dot{X}_\mu(\tau))}] - \int d^4y \, e \, \delta(y - X(\tau)) \dot{X}_\mu A^\mu(y) - \frac{1}{4} \int d^4y \, F^{\mu\nu}(y) F_{\mu\nu}(y) \\ &= \int [-(d\tau) \sqrt{(\dot{X}^\mu(\tau) \dot{X}_\mu(\tau))}] - \int (d\tau) \, e \, X_\mu \dot{X}^\mu(\tau) A^\mu(X(\tau)) - \frac{1}{4} \int d^4y \, F^{\mu\nu}(y) F_{\mu\nu}(y). \end{aligned} \quad (41)$$

Now we notice that the point particle and interaction sector depend on X_μ which are the dynamical variables. Indeed, when we consider the MS particle, X_μ need to be replaced by the physical x_μ variables

that obey the NC algebra. On the other hand, the argument y_μ in the pure Maxwell sector are just parameters and not dynamical variables which obviously are given by $A_\mu(y)$. Hence in the free Maxwell theory and resulting Maxwell equations there is no change due to the MS particle. The only change that will appear is in the interacting system via the source term in the Maxwell equation,

$$\partial^\mu F_{\mu\nu} = j_\nu(x), \quad j_\nu = ev_\nu \quad (42)$$

Now we can consider the Lienert-Wiechert potential for a moving charged MS particle. For the normal free charged particle this retarded potential is obtained from the inhomogeneous equation

$$\partial^\nu \partial_\nu A^\mu = j^\mu, \quad (43)$$

in Lorentz gauge $\partial_\mu A^\mu = 0$. This leads to the general solution

$$A^\mu(x) = A_0^\mu(x) + \int dx' G(x - x') j^\mu(x'), \quad (44)$$

where $A_0^\mu(x)$ satisfies the homogeneous equation $\partial^2 A_0^\mu(x) = 0$ (see for example [14]). Using the retarded form of the Green's Function $G(x - x')$ one finds the potential for a current j^μ ,

$$A^\mu(x) = A_0^\mu(x) + \frac{1}{4\pi} \int dx' \frac{\delta(x^0 - x'^0 - |\vec{x} - \vec{x}'|)}{|\vec{x} - \vec{x}'|} j^\mu(x'). \quad (45)$$

The form of current used by us is

$$j^\mu(x) = e \int d^4y \delta(y - x(\tau)) v_\mu(y). \quad (46)$$

Hence, recalling the argument presented above, for the retarded potential for the MS particle we need only introduce the form of MS current which in turn means we should use the form of velocity appropriate for the MS particle.

We return to our equations of motion valid to $O(1/\kappa)$:

$$\begin{aligned} \frac{dx^\alpha}{d\tau} &\sim -(1 + 3\frac{(\eta p)}{\kappa}) \frac{p^\alpha}{m_0} + \frac{e}{m_0 \kappa} (\eta F p) x^\alpha, \\ \frac{dp^\alpha}{d\tau} &\sim \frac{e}{m_0} (1 + \frac{(\eta p)}{\kappa}) F^{\alpha\beta} p_\beta + \frac{e}{m_0 \kappa} ((\eta F p) p^\alpha + 2m_0^2 F^{\alpha\beta} \eta_\beta). \end{aligned} \quad (47)$$

Again, to $O(e)$ and $O(1/\kappa)$, the above equations can be inverted to read

$$\frac{p^\alpha}{m_0} \sim -[(1 - 3\frac{(\eta p)}{\kappa}) \frac{dx^\alpha}{d\tau} + \frac{e}{\kappa} (\eta F \frac{dx}{d\tau}) x^\alpha]. \quad (48)$$

This motivates us to replace the velocity $(dx^\alpha)/(d\tau)$ in the charge current by the MS velocity

$$\frac{dx^\alpha}{d\tau} \rightarrow [(1 - 3\frac{(\eta p)}{\kappa}) \frac{dx^\alpha}{d\tau} + \frac{e}{\kappa} (\eta F \frac{dx}{d\tau}) x^\alpha]. \quad (49)$$

First of all we consider the case with no external field, $F_{\mu\nu} = 0$ so that the replacement is simply

$$\frac{dx^\alpha}{d\tau} \rightarrow (1 - 3\frac{W}{\kappa})\frac{dx^\alpha}{d\tau}, \quad (50)$$

where the energy W is computed from the MS dispersion relation (5),

$$W = [-\frac{m_0^2}{\kappa} + \sqrt{(\frac{m_0^2}{\kappa})^2 + (1 - \frac{m_0^2}{\kappa^2})(\vec{p}^2 + m_0^2)}]/(1 - \frac{m_0^2}{\kappa^2}). \quad (51)$$

To $O(\kappa)$ this simplifies to

$$W \sim \sqrt{\vec{p}^2 + m_0^2} - \frac{m_0^2}{\kappa} \quad (52)$$

with the rest energy being

$$W_0 \sim (1 - \frac{m_0}{\kappa})m_0.$$

Hence, for a MS particle at rest, the factor $(1 - 3\frac{W_0}{\kappa}) \sim (1 - 3\frac{m_0}{\kappa})$ modifies the Coulomb potential,

$$A_0 = \frac{e}{4\pi R}(1 - 3\frac{m_0}{\kappa}); \quad A_i = 0. \quad (53)$$

Defining $\vec{R} = \vec{x} - \vec{y}(\tau_0)$, $\vec{n} = \vec{R}/R$, $\vec{v} = d\vec{x}/dt$, we find the situation is much more involved for a moving MS particle because now the factor $(1 - 3\frac{W}{\kappa})$ is no longer a constant but depends on velocity,

$$A_0 = \frac{e}{4\pi R(1 - \vec{n} \cdot \vec{v})}(1 - 3\frac{W}{\kappa}); \quad \vec{A} = \frac{e\vec{v}}{4\pi R(1 - \vec{n} \cdot \vec{v})}(1 - 3\frac{W}{\kappa}). \quad (54)$$

However, probably more studies are needed to interpret our results for the MS particle induced retarded potential in the presence of external field $F_{\mu\nu}$ due to the presence of the x^α -dependent term in (49) in the expression for velocity. We will not pursue this analysis any further in the present publication.

VI. DISCUSSION, CONCLUSION AND FUTURE OUTLOOK

We will mainly study the results obtained in Section IV regarding the paths and trajectories of MS particles in external electromagnetic field. Let us demonstrate pictorially the κ -effects by way of diagrams. For the sake of comparison we have always provided a diagram for $1/\kappa = 0$ depicting the conventional particle. In the Figure 1 we plot the particle energy as a function of τ . To our approximation of $O(1/\kappa)$ the diagrams are not qualitatively different from the $1/\tau = 0$ (conventional particle) case but the energy increases more rapidly as $1/\kappa$ increases. Figure 2 shows the plot of x_1 vs. τ that is the coordinate in the direction of the external field. Once again due to the approximation involved all the curves show hyperbolic nature with the graphs with larger $1/\kappa$ more steeply curved. However in Figure 3, where we plot x_2 vs. τ , there appear qualitatively different paths for the MS particle in relation to the conventional

particle. The latter is a simple straight line as there is no acceleration along x_2 in the conventional particle case but for MS particle the hyperbolic nature dominates for later τ due to the κ -dependent acceleration term in x_2 .

In the second set of figures (Figures 4,5 and 6) we show the trajectories in the $x_1 - x_2$ plane for different choices of external parameters. In Figure 4, we plot the trajectories for different values of κ for fixed external conditions: constant values of $\epsilon \sim$ electric field E in x_1 -direction and constant $v_0 \sim$ initial energy of the particle. In Figure 5 we keep v_0 unchanged and consider the effect of varying ϵ on different κ -valued particles. Figure 6 shows the effect of different v_0 on different κ -valued particles where throughout ϵ is unaltered. Some common features of the trajectories are the following: Quite expectedly the effects are more pronounced as $1/\kappa$ increases, as is apparent from Figure 4. In both figures 5 and 6, we have compared the behavior of $1/\kappa = 0.1$ particle with $1/\kappa = 0$, the conventional particle. We notice that the κ -effects are more appreciable for larger values of the external parameters. This is also understandable since the external parameters are directly connected to the energy scales of the system and all κ -corrections involve the ratio of $(energy)/\kappa$. Lastly a general feature is that in all instances of fixed ϵ and v_0 the κ -particle trajectory bent more toward the x_2 -axis with respect to the conventional $\kappa = 0$ particle. This is because κ -corrections produce a non-trivial acceleration in the x_2 -direction even though the electric field is in x_1 -direction, in contrast to the conventional particle for which the velocity in x_2 direction remains constant. From the relation(40) also it is clear that the eccentricity Σ of the hyperbolic trajectory decreases as $1/\kappa$ increases.

Finally let us summarize our work. We have studied the behavior of an extended model of charged particle, pertaining to the Magueijo-Smolin form of particle in the Doubly Special Relativity framework. The free particle resides in a phase space with κ -Minkowski form of Non-Commutative symplectic structure. As we have argued in the paper, studies of the above free particle model carried out so far in the literature is indeed interesting and important but the free particle can not by itself provide a true picture. It is imperative to study the particle in the presence of non-trivial interactions as the interacting model can show the novelties as well as possible drawbacks (if there are any) of the new exotic particle models. In the present paper we have studied in detail the charged Magueijo-Smolin particle interacting with electromagnetic field, both external as well as dynamical, in a perturbative framework. As we have demonstrated there are non-trivial κ -corrections in the particle trajectories. We have briefly discussed the dynamical gauge fields as well and how the Lienard-Wiechert type of potentials get modified due to the κ -corrected source term. Deeper analysis is required in the latter area.

For future work we intend to take up dynamics of the gauge field more thoroughly in the κ -Minkowski framework, especially in $2 + 1$ -dimensions that is an area of recent interest [15]. It is interesting to note that $2 + 1$ -dimensional spacetime allows more general forms gauge kinetic term other than just the Maxwell term, such as the Chern-Simons term or a combination of both the Maxwell and Chern-Simons

term (that is the topologically massive gauge theory [16]). We will report on the MS particle dynamics in these forms of gauge kinetic terms in future publications.

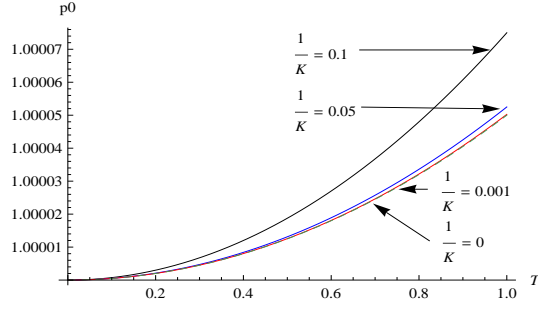


FIG. 1: Plot of p_0 vs. τ for fixed $\epsilon = 0.01, m = 1$ and different κ with $\frac{1}{\kappa} = 0$ for broken line.

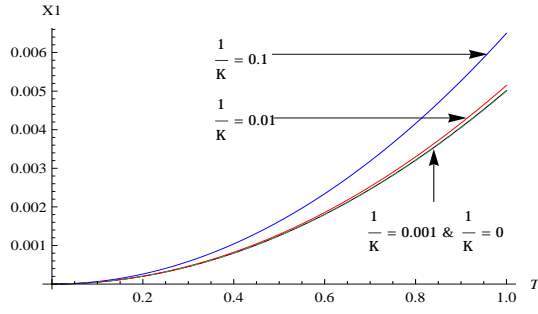


FIG. 2: Plot of x_1 vs. τ for fixed $\epsilon = 0.01, m = 1$ and different κ with $\frac{1}{\kappa} = 0$ for broken line.

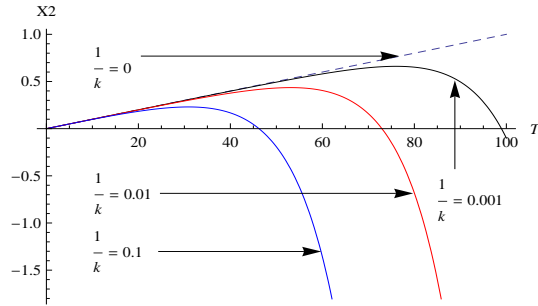


FIG. 3: Plot of x_2 vs. τ for fixed $\epsilon = 0.1, m = 1$ and different κ with $\frac{1}{\kappa} = 0$ for broken line.

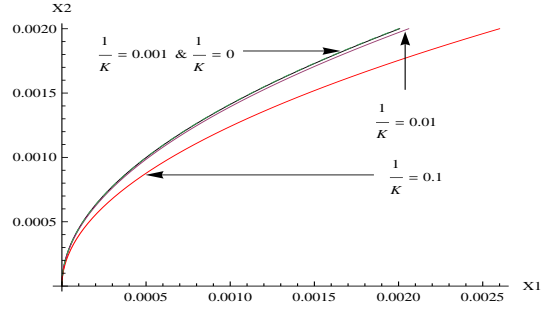


FIG. 4: Trajectory plot for different κ with $\epsilon = 0.001$, initial velocity $v_0 = 0.001$.

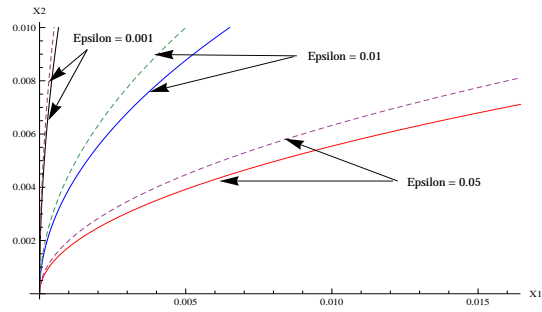


FIG. 5: Trajectory plot for fixed $v_0 = 0.01$ and different ϵ with $\frac{1}{\kappa} = 0.1$ (unbroken line) and $\frac{1}{\kappa} = 0$ (broken line).

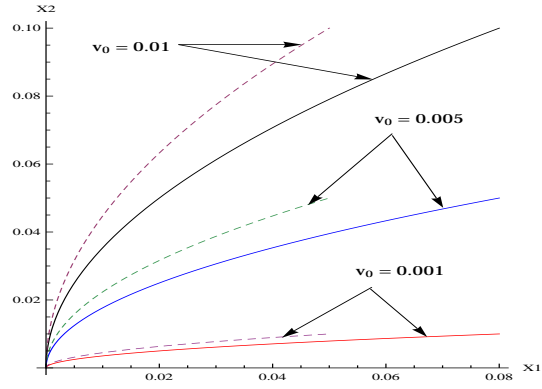


FIG. 6: Trajectory plot for fixed $\epsilon = 0.001$ and different v_0 with $\frac{1}{\kappa} = 0.2$ (unbroken line) and $\frac{1}{\kappa} = 0$ (broken line).

VII. APPENDIX

We provide some computational details for the solutions of the system of non-linear ordinary differential equation (29-32),

$$\frac{d^2 p_0}{d\tau^2} = \epsilon^2 \left[p_0 + \frac{1}{\kappa} (3p_0^2 + p_1^2 + 2m^2) \right], \quad \frac{d^2 p_1}{d\tau^2} = \epsilon^2 \left[p_1 + \frac{2}{\kappa} p_0 p_1 \right], \quad (55)$$

$$\frac{d^2 p_2}{d\tau^2} = -\frac{\epsilon^2}{\kappa} p_0 p_2, \quad \frac{d^2 p_3}{d\tau^2} = -\frac{\epsilon^2}{\kappa} p_0 p_3 \quad (56)$$

We solve it in a perturbative framework in the first non-trivial order in $1/\kappa$. Let us assume $p_{00}, p_{10}, p_{20}, p_{30}$ to be the solution of p_0, p_1, p_2, p_3 , for $\kappa \rightarrow \infty$, that is the conventional case. Now we expand p_μ about the conventional solutions $p_{\mu 0}$:

$$p_0 = p_{00} + \frac{c}{\kappa} p_{01}, \quad p_1 = p_{10} + \frac{d}{\kappa} p_{11}, \quad p_2 = p_{20} + \frac{g}{\kappa} p_{21}, \quad p_3 = p_{30} + \frac{h}{\kappa} p_{31} \quad (57)$$

Here c, d, g, h are constants to be evaluated later. If at $\tau = 0$ the initial velocity v_0 is along y -axis then initial condition for the above set of perturbation are

$$p_0(0) = m \Rightarrow p_{00}(0) = m, \quad p_{01}(0) = 0,$$

$$p_1(0) = 0 \Rightarrow p_{10}(0) = p_{11}(0) = 0,$$

$$p_2(0) = mv_0 \Rightarrow p_{20}(0) = mv_0, \quad p_{21}(0) = 0,$$

$$p_3(0) = 0 \Rightarrow p_{30}(0) = p_{31}(0) = 0 \quad (58)$$

Using (57) the above set of differential equation becomes

$$\frac{d^2 p_{00}}{d\tau^2} = \epsilon^2 p_{00}, \quad \frac{d^2 p_{10}}{d\tau^2} = \epsilon^2 p_{10}, \quad \frac{d^2 p_{20}}{d\tau^2} = 0, \quad \frac{d^2 p_{30}}{d\tau^2} = 0, \quad (59)$$

$$c \frac{d^2 p_{01}}{d\tau^2} = \epsilon^2 [c p_{01} + 3p_{00}^2 + p_{10}^2 + 2m^2], \quad (60)$$

$$d \frac{d^2 p_{11}}{d\tau^2} = \epsilon^2 [d p_{11} + 2p_{00} + p_{10}], \quad (61)$$

$$g \frac{d^2 p_{21}}{d\tau^2} = -\epsilon^2 p_{00} p_{20}, \quad h \frac{d^2 p_{31}}{d\tau^2} = -\epsilon^2 p_{00} p_{30}. \quad (62)$$

The solution of (59) are respectively

$$p_{00} = a_1 e^{\epsilon\tau} + a_2 e^{-\epsilon\tau}, \quad p_{10} = b_1 e^{\epsilon\tau} + b_2 e^{-\epsilon\tau} \quad (63)$$

$$p_{20} = c_1 + c_2 \tau, \quad p_{30} = d_1 + d_2 \tau \quad (64)$$

Now using (63) and (64) the solution of the differential equation (60),(61) and (62) are respectively

$$p_{01} = l_1 e^{\epsilon\tau} + l_2 e^{-\epsilon\tau} + \frac{3a_1^2 + b_1^2}{3c} e^{2\epsilon\tau} + \frac{3a_2^2 + b_2^2}{3c} e^{-2\epsilon\tau} - \frac{6a_1 a_2 + 2b_1 b_2 + 2m^2}{c} \quad (65)$$

$$p_{11} = m_1 e^{\epsilon\tau} + m_2 e^{-\epsilon\tau} + \frac{2a_1 b_1}{3D} e^{2\epsilon\tau} + \frac{2a_2 b_2}{3d} e^{-2\epsilon\tau} - \frac{2(a_1 b_2 + b_1 a_2)}{d} \quad (66)$$

$$p_{21} = -\frac{1}{g} \left[\left(a_1 c_1 - \frac{2a_1 c_2}{\epsilon} + \tau a_1 c_2 \right) e^{\epsilon\tau} + \left(a_2 c_1 + \frac{2a_2 c_2}{\epsilon} + \tau a_2 c_2 \right) e^{-\epsilon\tau} \right] + c' \tau + c'' \quad (67)$$

$$p_{31} = -\frac{1}{h} \left[\left(\tau - \frac{2}{\epsilon} \right) a_1 d_2 e^{\epsilon\tau} + \left(\tau + \frac{2}{\epsilon} \right) a_2 d_2 e^{-\epsilon\tau} \right] + c''' \tau + c'''' \quad (68)$$

Therefore

$$p_0 = \left(a_1 + \frac{cl_1}{\kappa} \right) e^{\epsilon\tau} + \left(a_2 + \frac{cl_2}{\kappa} \right) e^{-\epsilon\tau} + \frac{3a_1^2 + b_1^2}{3\kappa} e^{2\epsilon\tau} + \frac{3a_2^2 + b_2^2}{3\kappa} e^{-2\epsilon\tau} - \frac{6a_1 a_2 + 2b_1 b_2 + 2m^2}{c\kappa} \quad (69)$$

$$p_1 = \left(b_1 + \frac{dm_1}{\kappa} \right) e^{\epsilon\tau} + \left(b_2 + \frac{dm_2}{\kappa} \right) e^{-\epsilon\tau} + \frac{2a_1 b_1}{3\kappa} e^{2\epsilon\tau} + \frac{2a_2 b_2}{3\kappa} e^{-2\epsilon\tau} - \frac{2(a_1 b_2 + b_1 a_2)}{\kappa} \quad (70)$$

$$p_2 = c_1 + c_2 \tau - \frac{1}{\kappa} \left[\left(a_1 c_1 + \tau a_1 c_2 - \frac{2a_1 c_2}{\epsilon} \right) e^{\epsilon\tau} + \left(a_2 c_1 + \tau a_2 c_2 + \frac{2a_2 c_2}{\epsilon} \right) e^{-\epsilon\tau} \right] + \frac{c' g \tau}{\kappa} + \frac{c'' g}{\kappa} \quad (71)$$

$$p_3 = d_2 \tau - \frac{1}{\kappa} \left[\left(\tau - \frac{2}{\epsilon} \right) a_1 d_2 e^{\epsilon\tau} + \left(\tau + \frac{2}{\epsilon} \right) a_2 d_2 e^{-\epsilon\tau} \right] + \frac{c''' h \tau}{\kappa} + \frac{c'''' h}{\kappa} \quad (72)$$

All the arbitrary constants introduced above are solved by checking consistency of the set of solutions leading to the result given below:

$$a_1 = a_2 = \frac{m}{2}, \quad b_1 = -b_2 = \frac{m}{2}, \quad cl_1 = cl_2 = \frac{m^2}{6} + m^2, \quad dm_1 = -dm_2 = \frac{m^2}{6} + m^2, \\ c_1 = mv_0, \quad c_2 = 0, \quad c' = 0, \quad c'' = \frac{c_1 m}{g}, \quad d_1 = 0, \quad d_2 = 0, \quad c''' = 0, \quad c'''' = 0 \quad (73)$$

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